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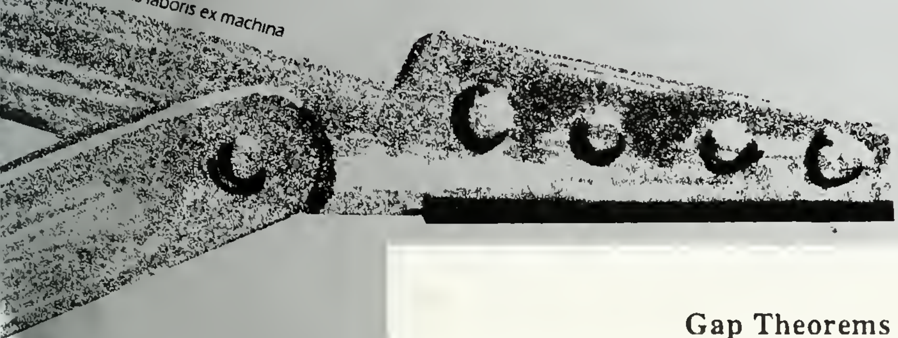

Gap Theorems

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Jiawei Hong

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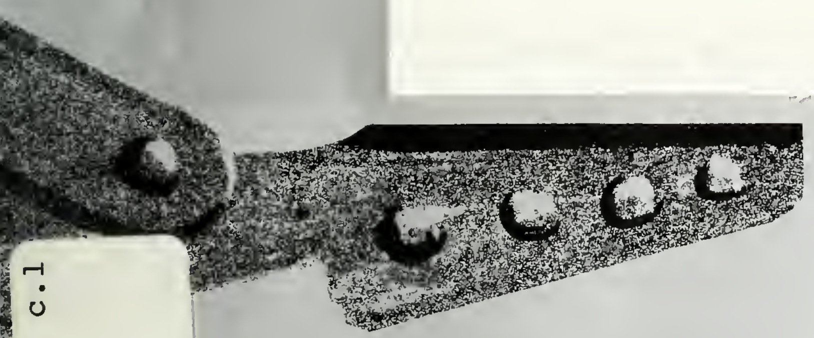

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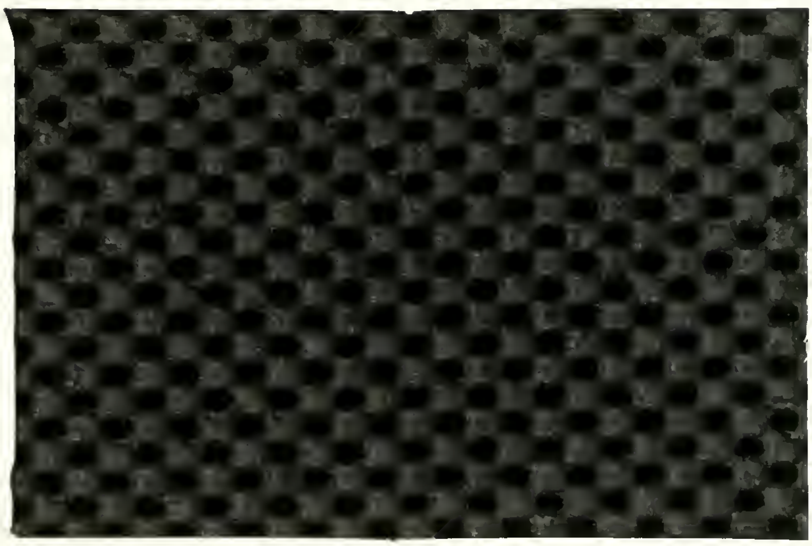

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Gap Theorems

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Abstract. In this paper we improve the gap theorem developed in the author's work **proving by example and gap theorems** [1], and prove some new general gap theorems to show that there are full of gaps in continuous mathematics.

1. General gap theorems in mathematics.

In continuous mathematics, there are many gap theorems which connect continuous mathematics with discrete mathematics. The following are some very general gap theorems.

Theorem 1. Assume that $g_n>0$, $n=1,2,3,...$ is a sequence (the gap) satisfying $\sum_{n=1}^{\infty} g_n<\infty$, and $a_n \in [0,1]$, $n=1,2,3,...$ is an arbitrary sequence. Then almost every point $x \in [0,1]$ satisfies $|x-a_n| \geq g_n$ for sufficiently large n. In other words, the Lebesgue measurement of the following set S, $|S|$, is 1.

$$S=\{s \in [0,1] \mid \text{for } \textit{sufficiently large } n,\ |x-a_n| \geq g_n\}$$

Proof. Define

$$S_i=\{x \in [0,1] \mid \text{for } \textit{more than } i \textit{ indices } j,\ |x-a_j|<g_j\}$$

$$T_i=\{x \in [0,1] \mid \textit{there exists } j \geq i,\ |x-a_j|<g_j\}.$$

Obviously, $|T_i| \leq \sum_{i}^{\infty} 2g_j \to 0 \ \ (i \to \infty)$.

If $x \in S_i$ then thee exists $j \geq i$ such that $|x-a_j|<g_j$, and therefore $x \in T_i$. So we have $S_i \subseteq T_i$. The complement set of S, $\bar{S}$, is contained in S_i: $\bar{S} \subseteq S_i \subseteq T_i$.

$$|\bar{S}| \leq |T_i| \to 0$$

Therefore $|\bar{S}|=0$ and $|S|=1$.

Theorem 2. Assume that there is an arbitrary mapping $\{0,1\}^* \to [0,1]$ (a representation method) which represents some real numbers in $[0,1]$ by some binary strings. Then almost every point $x \in [0,1]$ satisfies $|x-v| \geq 1/2^n n \log^2 n = 2^{-n-\log n-2\log\log n}$ (the gap) for sufficiently large n, where v is any value represented by a binary string of length $\leq n$. In other words, the

measurement of the following set is 1:

$S = \{x \in [0,1] \mid$ for *sufficiently large* n and *all* v *represented by a binary string of length* $\leq n$, $|x - v| \geq 1/2^n n \log^2 n\}$.

Proof. Number the binary strings so that the shorter is numbered earlier. Use the same number to number the binary string and the value represented by the binary string. The total number of binary strings of length $\leq n$ is $\leq 2^{n+1}$. Consider the gap $g_m = 4/m \log m \log^2 \log m$. If a_m is a value numbered m and represented by a binary string of length $\leq n$, then $m \leq 2^{n+1}$. Since $\sum_{m=1}^{\infty} g_m < \infty$, by Theorem 1 we know it is true for almost every x that

$$|x - a_m| \geq g_m \geq \frac{4}{2^{n+1}(n+1)\log^2(n+1)} \geq \frac{1}{2^n n \log^2 n} = 2^{-n - \log n - 2\log\log n}$$

for sufficiently large n.

The above two theorems can not be improved substantially, because the following two results:

Theorem 3. If $\sum_{i=1}^{\infty} g_i = \infty$, then there is a sequence a_n, $n = 1, 2, \ldots$ such that the following set is empty:

$S = \{x \in [0,1] \mid$ for *sufficiently large* n, $|x - a_n| \geq g_n\}$.

Proof. Consider the sequence

$$a_n = \sum_{i=1}^{n} g_i \quad (mod\ 1).$$

Theorem 4. There is a way to represent real numbers by binary strings such that the following set is empty. This means that there is no gap of size 2^{-n} at any point:

$S = \{x \in [0,1] \mid$ for *sufficiently large* n and *all* v *represented by a binary string of length* n, $|x - v| \geq 2^{-n}\}$.

Proof. Consider the binary fraction in $[0,1]$.

2. Gap theorems in algebraic computation.

The gap theorems in last section are very general, but not very useful because we only know that the gap exists at almost every point but we do not know where it exists. For example we do not know whether there is a gap at point 0. In this section, we consider algebraic computation and show some gap theorems which guarantees that a gap exists at some

interesting points (for example, at 0).

Let $F(x_1,x_2,...x_i)$ be polynomials in $x_1,x_2,...,x_i$ with integer coefficients. The total sum of absolute values of all coefficients in F is called its weight. The highest total degree of $x_1,x_2,...,x_i$ in any term of F is called its degree. It is not hard to see that the degree of the sum of two polynomials is the maximum degree of these two polynomials, the degree of the product of two polynomials is the sum of their degrees, and the weight of the sum or product of two polynomials is bounded by the sum or product of their weights.

Let $F_i(x_1,x_2,...,x_i)$, $i=1,2,3,...$, be polynomials in $x_1,x_2,...,x_i$ with integer coefficients of total degree bounded by d and weight bounded by w. According to the descending order of the last variable, x_i, we denote the leading coefficient of F_i by $L_i(x_1,x_2,...,x_{i-1})$. Let $\alpha_1,\alpha_2,\alpha_3,...$ be algebraic numbers satisfying

$$F_i(\alpha_1,\alpha_2, \ldots ,\alpha_i)=0$$

$$L_i(\alpha_1,\alpha_2, \ldots ,\alpha_{i-1})\neq 0, \quad i=1,2,3,... \qquad (1).$$

In [1], it is proved that if α_i is not 0 then it can not be too small. In more details, the following theorem was proved:

Theorem 5. There is a constant $c=c(d,w)$, depending on d and w, such that $\alpha_i \neq 0$ implies $|\alpha_i|\geq 2^{-c^i}$.

Now we know that $c(d,w)$ is in fact a polynomial of d and $\log w$, as shown by the following theorem:

Theorem 6. In the above theorem,

$$c=d(d+1)^2\log w.$$

Theorem 7. If α_i is not an integer, then for any integer N we have

$$|\alpha_i - N| \geq w^{-(d(d+1)^3)^i}.$$

The proofs of Theorems 6 and 7 will be given in the next section. In [2], the following gap theorem is proved:

Theorem 8. Starting from 1, if we only use $+,-,*,/$ and $\sqrt{\ }$ in our computation, then the fastest convergence rate we can have is the quadratic convergence, no matter how precise our computation is. In other words, there is a constant $c<1$ such that if we start from 1, only use instructions $+,-,*,/,\sqrt{\ }$, the total number of instructions used is n and $\alpha\neq 0$ is the final number we obtained, then

$$|\alpha|\geq c^{2^n}.$$

3. The proofs of Theorem 6 and theorem 7.

Now we concentrate to the proofs of Theorem 6 and 7. Without loss of generality, we can assume that the highest power of x_i in F_i is d. In the following, we express α_i as a quotient of two algebraic integers, $\alpha_i = y_i/z_i$. Consider the first equation

$$F_1(x_1) = L_1 x_1^d + \dots + C_1 = 0, \quad L_1 \neq 0.$$

It equivalents to

$$L_1^d x_1^d + \dots + L_1^{d-1} C_1 = 0$$

or

$$f_1 = y_1^d + \dots + L_1^{d-1} C_1 = 0$$

where $y_1 = L_1 x_1$. The equation is now a monic one, therefore any solution for y_1 is an algebraic integer. Set $z_1 = L_1$. Then we have

$$y_1 = L_1 x_1,$$

$$z_1 = L_1 \neq 0,$$

$$\alpha_1 = y_1 / z_1.$$

Now consider the second equation

$$F_2(\alpha_1, x_2) = L_2(\alpha_1) x_2^d + \dots + C_2(\alpha_1) = 0.$$

To make the coefficients integral, we multiply it by z_1^d:

$$z_1^d L_2 x_2^d + \dots + z_1^d C_2 = 0.$$

Since the degree of α_1 in each coefficient of F_2 is bounded by d, the coefficient in the above equation can be expressed as a polynomial in y_1 with rational integer coefficients. Its degree is bounded by d. The above equation in turn equivalents to

$$(z_1^d L_2)^d x_2^d + \dots + (z_1^d L_2)^{d-1} z_1^d C_2 = 0$$

or

$$f_2 = y_2^d + \dots + (z_1^d L_2)^{d-1} z_1^d C_2 = 0,$$

where

$$y_2 = z_1^d L_2 x_2,$$

$$z_2 = z_1^d L_2 \neq 0,$$

$$\alpha_2 = y_2 / z_2.$$

Generally, consider the i-th equation

$$F_i(\alpha_1,\alpha_2, \ldots ,\alpha_{i-1},x_i)=L_i(\alpha_1,\alpha_2, \ldots ,\alpha_{i-1})x_i^d+\ldots+C_i(\alpha_1,\alpha_2, \ldots ,\alpha_{i-1})=0.$$

We multiply it by $(z_1z_2\ldots z_{i-1})^d$. Then the coefficients of the following equivalent equation will be integral:

$$(z_1z_2\ldots z_{i-1})^dL_ix_i^d+\ldots+(z_1z_2\ldots z_{i-1})^dC_i=0.$$

It equivalents to

$$((z_1z_2\ldots z_{i-1})^dL_i)^dx_i^d+\ldots+((z_1z_2\ldots z_{i-1})^dL_i)^{d-1}(z_1z_2\ldots z_{i-1})^dC_i=0$$

or

$$f_i=y_i^d+..+((z_1z_2\ldots z_{i-1})^dL_i)^{d-1}(z_1z_2\ldots z_{i-1})^dC_i=0,$$

where

$$y_i=(z_1z_2\ldots z_{i-1})^dL_ix_i$$

$$z_i=(z_1z_2\ldots z_{i-1})^dL_i\neq 0$$

$$\alpha_i=y_i/z_i.$$

Since each α_j in the coefficients of F_i has degree $\leq d$, each coefficient in the above equation can be expressed as a polynomial in $y_1,y_2,\ldots,y_i$ with rational integer coefficients. So does z_i.

We are going to figure out how complicated these y_i and z_i are, i.e., to give some upper bounds for their weights and degrees (as polynomials of $y_1,y_2,\ldots,y_i$). The following are some simple analysis.

Lemma 1. The weight of z_i, $w(z_i)$, is bounded by (the argument z_i is considered as an expression, not a value)

$$w(z_i)\leq w^{(d+1)^{i-1}}.$$

Proof. Obviously, $z_1=L_1$, therefore $w(z_1)\leq w$. This is the basis. Inductively, since $z_{i+1}=(z_1z_2\ldots z_i)^dL_{i+1}$, we have

$$w(z_{i+1})\leq(w(z_1)\ldots w(z_i))^dw(L_{i+1})$$

$$\leq w^{d(1+(d+1)+(d+1)^2+\ldots+(d+1)^{i-1})+1}$$

$$=w^{(d+1)^i}.$$

Lemma 2. The weight of f_i is bounded by

$$w(f_i)\leq w^{d(d+1)^{i-1}}.$$

Proof. Obviously, $w(f_1)\leq w(L_1)^{d-1}\leq w^d$, the basis is true. Inductively, since f_{i+1} is obtained

from F_{i+1} by multiplying $z_{i+1}^{d-1}(z_1z_2...z_i)^d$, we have

$$w(f_{i+1}) \le (w(z_{i+1}))^{d-1}(w(z_1)w(z_2)...w(z_i))^d.w$$
$$\le w^{(d+1)^i(d-1)+d(1+(d+1)+(d+1)^2+...+(d+1)^{i-1})+1}$$
$$\le w^{d(d+1)^i}.$$

In the following, we use $f << y_1^{d_1}y_2^{d_2} \cdots y_i^{d_i}$ to mean that in polynomial f, the degree of y_1 is bounded by d_1, the degree of y_2 is bounded by d_2,..., and so on.

Lemma 3. $z_i << y_1^{d(d+1)^{i-2}}y_2^{d(d+1)^{i-3}}...y_{i-1}^d$.

Proof. Since $z_1 = L_1$, a constant, and $z_2 = z_1^dL_2$, we know that the degree of y_1 in z_2 is bounded by d. Therefore $z_2 << y_1^d$. This is the basis. Inductively, since

$$z_{i+1} = (z_1z_2...z_i)^dL_{i+1},$$

we have

$$z_{i+1} << [(y_1^d)(y_1^{d(d+1)}y_2^d)(y_1^{d(d+1)^2}y_2^{d(d+1)}y_3^d)...$$
$$(y_1^{d(d+1)^{i-2}}y_2^{d(d+1)^{i-3}} \cdots y_{i-1}^d)]^d(y_1y_2...y_i)^d$$
$$= y_1^{d+d^2+d^2(d+1)+d^2(d+1)^2+...+d^2(d+1)^{i-2}}y_2^{d+d^2+d^2(d+1)+...+d^2(d+1)^{i-3}}...y_i^d$$
$$= y_1^{d(d+1)^{i-1}}y_2^{d(d+1)^{i-2}}...y_i^d.$$

Lemma 4. If C is a coefficient in f_i, then

$$C << y_1^{d^2(d+1)^{i-2}}y_2^{d^2(d+1)^{i-3}}...y_{i-1}^{d^2}.$$

Proof. The lemma is obviously true for $i = 1,2$. Inductively, we consider for example the degree in the last coefficient (the other terms have the same bound), which is

$$C = z_{i+1}^{d-1}(z_1z_2 \cdots z_i)^dC_{i+1}.$$

Therefore,

$$C << (y_1^{d(d-1)^{i-1}}y_2^{d(d-1)^{i-2}}...y_i^d)^{d-1}$$
$$(y_1^{d+d(d+1)+...+d(d+1)^{i-2}}y_2^{d+d(d+1)+...+d(d+1)^{i-3}}...y_{i-1}^d)^d(y_1y_2...y_i)^d$$
$$= y_1^{d^2(d+1)^{i-1}}y_2^{d^2(d+1)^{i-2}} \cdots y_i^{d^2}.$$

Now we can bound the absolute value of y_i from above.

Lemma 5. $|y_i| \le w^{d(d+1)^{2i-2}}$.

Proof. With out loss of generality, we can assume that $|y_i| \geq 0$. Then the absolute value of a root of a monic algebraic equation is bounded by the sum of the absolute values of the coefficients. Since the total weight of f_{i-1} is bounded by $w^{d(d+1)^i}$ and the degree of $y_1, y_2, \ldots y_i$ in each coefficient is bounded by $y_1^{d(d+1)^{i-1}} y_2^{d(d+1)^{i-2}} \cdots y_{i-1}^{d}$, we can prove the lemma inductively.

For $i=1$, $|y_i| \leq w(f_1) \leq w^d$, the lemma is true. Inductively,

$$|y_{i+1}| \leq w(f_{i+1}) y_1^{d^2(d+1)^{i-1}} y_2^{d^2(d+1)^{i-2}} \ldots y_{i-1}^{d^2}$$

$$\leq w^{d(d+1)^i} (w^{d^3(d+1)^{i-1}} w^{d^3(d+1)^i} \cdots w^{d^3(d+1)^{2i-2}})$$

$$\leq w^{d(d+1)^{i-1}(1+d+d^2+d^2(d+1)+d^2(d+1)^2+\ldots+d^2(d+1)^i)}$$

$$< w^{d(d+1)^{i-1}(d+1)^{i+1}}$$

$$= w^{d(d+1)^{2i}}.$$

Lemma 6. $|y_i| \geq w^{-d(d+1)^{2i-2}(d^i-1)}$.

Proof. Consider all conjugates of y_1, all conjugates of y_2, ..., all conjugates of y_i. There are at most d^i many conjugates of y_i, denoted by $y_i = \beta_1, \beta_2, \ldots, \beta_J$, $J \leq d^i$. The product of all these conjugates is a rational non-zero integer. Therefore its absolute value must be at least 1:

$$\left|\prod_{j=1}^{J} \beta_j\right| \geq 1.$$

Thus, by Lemma 5,

$$|y_i| \geq 1/(w^{d(d+1)^{2i-2}})^{d^i-1}.$$

Lemma 7. $|z_i| \leq w^{d(d+1)^{2i-3}+(d+1)^{i-2}}$.

Proof. By Lemmas 1,3,5,

$$|z_i| \leq w(z_i) y_1^{d(d+1)^{i-2}} y_2^{d(d+1)^{i-3}} \ldots y_{i-1}^{d}$$

$$\leq w^{(d+1)^{i-1}} [w^{d^2(d+1)^{i-2}} w^{d^2(d+1)^{i-1}} \ldots w^{d^2(d+1)^{2i-4}}]$$

$$= w^{(d+1)^{i-1} + d(d+1)^{2i-3} - d(d+1)^{i-2}}$$

$$= w^{d(d+1)^{2i-3}+(d+1)^{i-2}}.$$

Now we can prove our Theorem 6 as follows:

Proof of Theorem 6.

$$|\boldsymbol{\alpha}_i| = |y_i/z_i| \geq w^{-d(d+1)^{2i-2}(d^i-1)-d(d+1)^{2i-3}-(d+1)^{i-2}}$$

$$\geq w^{-d^{i+1}(d+1)^{2i-2}} \geq w^{-(d(d+1)^2)^i}.$$

Lemma 8.

$$|z_i| \geq w^{-(d(d+1)^{2i-3}+(d+1)^{i-2})(d^i-1)}$$

$$|\alpha_i| < w^{(d+1)^{2i-2}d^i}.$$

Proof. $|z_i|$ can be bounded using the same method as in Lemma 6. Therefore we have

$$|\alpha_i| \leq w^{d(d+1)^{2i-2}+(d(d+1)^{2i-3}+(d+1)^{i-2})(d^i-1)} < w^{(d+1)^{2i-2}d^i}.$$

Proof of Theorem 7. We can assume that $|\alpha_i - N| \leq 1$. Then by Lemma 8 we have

$$|N| \leq w^{(d+1)^{2i-2}d^i}$$

$$|y_i - Nz_i| \leq w^{d(d+1)^{2i-2}} + w^{(d+1)^{2i-2}d^i+d(d+1)^{2i-3}+(d+1)^{i-2}} \leq w^{(d+1)^{3i-2}}.$$

Therefore, by the same method as in Lemma 6, we have

$$|y_i - Nz_i| \geq w^{(d+1)^{3i-2}(d^i-1)}.$$

Thus,

$$|\alpha_i - N| = |y_i - Nz_i| / |z_i| \geq w^{-d^i(d+1)^{3i-2}} \geq w^{-(d(d+1)^3)^i}.$$

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